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On weighted extensions of Cauchy's means

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Abstract

We show that every Cauchy mean in $(0,\infty)$ can be embedded into two parameter family of weighted means. Some basic properties and examples are presented. A functional equation which appears in the problem of symmetry of these means is considered. As an application a natural extension of Stolarsky's means is obtained and a two parameter subclass of weighted power means is determined.

Keywords: Cauchy mean; Lagrangean mean; Stolarsky mean; Weighted quasi-arithmetic mean; Weighted power mean; Weighted extension of Cauchy mean; Functional equation

1. Introduction

A function $M : I \times I \to \mathbb{R}$ is called a *mean* in an interval $I \subset \mathbb{R}$ if

$$\min(x, y) \leqslant M(x, y) \leqslant \max(x, y), \qquad x, y \in I.$$

A mean M is called strict if these inequalities are strict for all $x, y \in I$, $x \neq y$; and symmetric if $M(x, y) = M(y, x)$ for all $x, y \in I$. A function $M : I \times I \to \mathbb{R}$ is called *increasing* if it is increasing with respect to each of the variables. It is obvious that an increasing function M is a mean iff it is reflexive, i.e. if $M(x, x) = x$ for all $x \in I$. A mean $M : (0,\infty)^2 \to (0,\infty)$ is called *homogeneous* if

$$M(tx, ty) = tM(x, y), \qquad t, x, y > 0.$$

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It is well known that for the differentiable functions $f, g : I \to \mathbb{R}$ such that $g' \neq 0$ and $\frac{f'}{g'}$ is invertible, applying the mean-value theorem, one can define the *Cauchy mean* $D^{[f,g]} : I^2 \to I$. For $g := \mathrm{id}|_I$ this mean reduces to a *Lagrangean mean*. In Section 2 we remark that every Cauchy mean is *g-conjugate* to the Lagrangean mean $L^{[f \circ g^{-1}]}$. In Section 4 we show that every Cauchy mean in $I = (0, \infty)$ can be embedded in a natural way into two parameter family of means $D_{a,b}^{[f,g]}$ (Theorem 2). Some basic properties of these means, examples and discussion of the possibility to determine the effective formula for these means are presented. To prove that $D_{a,b}^{[f,g]}$ is symmetric iff $a = b$, in Section 3 we deal with the functional equation

$$\frac{F(x) - F(y)}{G(x) - G(y)} = \frac{F(x+1) - F(y-1)}{G(x+1) - G(y-1)},$$

where F and G are the unknown functions (Theorem 1). In Section 5 we apply Theorem 2 to imbed the two-parameter family of the Stolarsky means $\{E^{[p,q]}\colon\ p, q \in \mathbb{R}\}$ into a three parameter family $\{E_t^{[p,q]}\colon\ p, q \in \mathbb{R},\ t > 0\}$ which can be treated as the weighted means. In Section 6 we determine those of the means $E_t^{[p,q]}$ which are weighted and quasi-arithmetic (Theorem 4).

2. Cauchy mean is conjugate to a Lagrangean one

If the functions $f, g : I \to \mathbb{R}$ are differentiable in an interval I, $g' \neq 0$ and $\frac{f'}{g'}$ is one-to-one, then the function $D^{[f,g]} : I^2 \to I$,

$$D^{[f,g]} := \begin{cases} \left(\frac{f'}{g'}\right)^{-1}\left(\frac{f(x)-f(y)}{g(x)-g(y)}\right), & x \neq y, \\ x, & x = y, \end{cases}$$

is correctly defined, and it is a strict symmetric mean. $D^{[f,g]}$ is called a *Cauchy mean* generated by f and g.

Remark 1. Let $I \subset \mathbb{R}$ be an interval and suppose that $f, g : I \to \mathbb{R}$ are differentiable and $g' \neq 0$. If $\frac{f'}{g'} : I \to \mathbb{R}$ is one-to-one, then it is strictly monotonic and continuous.

Proof. The assumption that $g' \neq 0$ and the Darboux property of derivative imply that g' is of a constant sign in I and, consequently, g is strictly monotonic and continuous. Consequently, g^{-1}, the inverse function of g, is differentiable on $g(I)$ and strictly monotonic. From the identity $\frac{f'}{g'} \circ g^{-1} = (f \circ g^{-1})'$, the Darboux property of derivative and the assumption that $\frac{f'}{g'}$ is one-to-one we infer that $\frac{f'}{g'}$ is continuous and strictly monotonic. $\square$

Remark 2. Suppose that $f, g : I \to \mathbb{R}$ are differentiable in an interval I, $g' \neq 0$ and $\frac{f'}{g'}$ is one-to-one. According to Remark 1, the function $\frac{f'}{g'}$ is continuous and strictly monotonic.

If $f' \neq 0$ in I then the mean $D^{[g,f]}$ is uniquely defined and, the relations

$$\frac{f'}{g'}\left(D^{[f,g]}(x, y)\right) = \frac{f(x) - f(y)}{g(x) - g(y)}, \qquad \frac{g(x) - g(y)}{f(x) - f(y)} = \frac{g'}{f'}\left(D^{[g,f]}(x, y)\right)$$

imply that

$$D^{[g,f]} = D^{[f,g]}.$$

In the opposite case, by the strict monotonicity of $\frac{f'}{g'}$, there is only one point x_0 in I such that $f'(x_0) = 0$. Note that for all $x, y \in I$, such that $f(x) \neq f(y)$, the above relations allow define

$D^{[g,f]}(x,y)$ and conclude that $D^{[g,f]}(x,y) = D^{[f,g]}(x,y)$. For all $x, y \in I$, $x \neq y$ such that $f(x) = f(y)$ we can define $D^{[g,f]}(x,y)$ by

$$D^{[g,f]}(x,y) := \lim_{(u,v)\to(x,y)} D^{[f,g]}(u,v).$$

Thus, if $D^{[f,g]}$ is defined, then so is the mean $D^{[g,f]}$, and $D^{[g,f]} = D^{[f,g]}$.

If $g = \mathrm{id}|_I$ then f' is invertible and $D^{[f,g]}$ reduces to the Lagrangean mean $L^{[f]} : I^2 \to I$,

$$L^{[f]}(x,y) := \begin{cases} (f')^{-1}\left(\frac{f(x)-f(y)}{x-y}\right), & x \neq y, \\ x, & x = y, \end{cases}$$

generated by the function f.

Let us note the following:

Proposition 1. *Let $f, g : I \to \mathbb{R}$ be differentiable, $g' \neq 0$ and $\frac{f'}{g'}$ one-to-one, then*

$$D^{[f,g]}(x,y) = g^{-1}\left(L^{[f\circ g^{-1}]}\left(g(x), g(y)\right)\right), \quad x, y \in I.$$

Proof. By the Lagrange mean-value theorem, for $u, v \in g(I)$, $u \neq v$, we have

$$\frac{f \circ g^{-1}(u) - f \circ g^{-1}(v)}{u - v} = \left(f \circ g^{-1}\right)'\left(L^{[f\circ g^{-1}]}(u,v)\right) = \left(\frac{f'}{g'}\right) \circ g^{-1}\left(L^{[f\circ g^{-1}]}(u,v)\right),$$

whence

$$L^{[f\circ g^{-1}]}(u,v) = g \circ \left(\frac{f'}{g'}\right)^{-1}\left(\frac{f \circ g^{-1}(u) - f \circ g^{-1}(v)}{u - v}\right).$$

Taking $u = g(x)$, $v = g(y)$ for $x, y \in I$, $x \neq y$, we hence get

$$g^{-1}\left(L^{[f\circ g^{-1}]}\left(g(x), g(y)\right)\right) = \left(\frac{f'}{g'}\right)^{-1}\left(\frac{f(x) - f(y)}{g(x) - g(y)}\right) = D^{[f,g]}(x,y),$$

which was to be shown. □

Remark 3. Let $M : I \times I \to I$ be a mean in an interval $I \subset \mathbb{R}$. If $J \subset \mathbb{R}$ is an interval and $\varphi : J \to I$ continuous, strictly increasing and onto, then the function

$$J \times J \ni (x,y) \to \varphi^{-1}\left(M\left(\varphi(x), \varphi(y)\right)\right)$$

is a mean in J and it is called φ-conjugate of the mean M.

Remark 4. According to this proposition, every Cauchy mean $D^{[f,g]}$ is g-conjugate of the Lagrangean mean $L^{[f\circ g^{-1}]}$ generated by the function $f \circ g^{-1}$.

3. A functional equation

Theorem 1. *Suppose that $F, G : \mathbb{R} \to \mathbb{R}$ are continuous and G is strictly monotonic. If F and G satisfy the functional equation*

$$\frac{F(x)-F(y)}{G(x)-G(y)}=\frac{F(x+1)-F(y-1)}{G(x+1)-G(y-1)}, \quad x,y\in\mathbb{R},\ x\neq y\neq x+2, \tag{1}$$

then there are $A, B \in \mathbb{R}$ *such that* $F = AG + B$.

Proof. Suppose that $F, G:\mathbb{R}\to\mathbb{R}$ satisfy Eq. (1). The functions $A[F-F(0)]$ and $G-G(0)$, where $A\neq 0$ is an arbitrary real constant, also satisfy this equation. Therefore, without any loss of generality, we can assume that F is not constant and

$$F(0)=G(0)=0, \qquad F(1)=G(1)=1.$$

Thus it is enough to show that $F=G$ or, equivalently, that the function $\varphi: G(\mathbb{R})\to\mathbb{R}$ defined by

$$\varphi := F\circ G^{-1}$$

is the identity. For an indirect argument suppose that $\varphi\neq \mathrm{id}|_{G(\mathbb{R})}$ and put

$$W := \{x\in G(\mathbb{R}):\ \varphi(x)\neq x\}.$$

Note that W is open in $\mathbb{R}$ and $0, 1\notin W$ that is

$$\varphi(0)=0, \qquad \varphi(1)=1.$$

From (1) we get

$$\frac{\alpha(x)-\beta(y)}{\gamma(x)-\delta(y)}=\frac{\varphi(x)-\varphi(y)}{x-y}, \tag{2}$$

for all $x, y\in G(\mathbb{R})$; $x\neq y\neq G(G^{-1}(x)+2)$, where

$$\alpha(x) := F\big(G^{-1}(x)+1\big), \qquad \beta(x) := F\big(G^{-1}(x)-1\big),$$
$$\gamma(x) := G\big(G^{-1}(x)+1\big), \qquad \delta(x) := G\big(G^{-1}(x)-1\big)$$

for $x\in G(\mathbb{R})$.

Setting $y=0$ and then $x=0$ in (2), and taking into account that $\varphi(0)=0$, we obtain, for all $x, y\in G(\mathbb{R})\setminus\{0\}$,

$$\alpha(x)=\frac{\varphi(x)}{x}\big(\gamma(x)-d\big)+b, \qquad \beta(y)=\frac{\varphi(y)}{y}\big(\delta(y)-c\big)+a,$$

where

$$a := \alpha(0), \qquad b := \beta(0), \qquad c := \gamma(0), \qquad d := \delta(0),$$

whence, by (2),

$$\frac{\frac{\varphi(x)}{x}(\gamma(x)-d)+b-\frac{\varphi(y)}{y}(\delta(y)-c)-a}{\gamma(x)-\delta(y)}=\frac{\varphi(x)-\varphi(y)}{x-y}, \tag{3}$$

for all $x, y\in G(\mathbb{R})\setminus\{0\}$; $x\neq y\neq G(G^{-1}(x)+2)$.

Setting here first $y=1$ and then $x=1$, and taking into account that $\varphi(1)=1$ we obtain, respectively,

$$\gamma(x)=\frac{[(d-q)x-d]\varphi(x)+(a-b-c+q)x^2+(b+c-a)x}{x-\varphi(x)}, \quad x\in W, \tag{4}$$

and

$$\delta(y)=\frac{[(c-p)y-c]\varphi(y)+(b-a-d+p)y^2+(a+d-b)y}{y-\varphi(y)},\quad y\in W,\tag{5}$$

where

$$p:=\gamma(1),\qquad q:=\delta(1).$$

Replacing $\gamma(x)$ and $\delta(y)$ in (3) by the right-hand sides of formulas (4) and (5), respectively, we obtain

$$\begin{aligned}
&\Big(\big(y-\varphi(y)\big)\varphi(x)\big[A\varphi(x)+Bx+C\big]-\big(x-\varphi(x)\big)\varphi(y)\big[D\varphi(y)+Ey+H\big]\\
&\qquad+K\big(x-\varphi(x)\big)\big(y-\varphi(y)\big)\Big)\\
&\qquad\times\Big(\big(y-\varphi(y)\big)\big[(Ax-d)\varphi(x)+Bx^2+Mx\big]\\
&\qquad-\big(x-\varphi(x)\big)\big[(Dy-c)\varphi(y)+Ey^2+Ny\big]\Big)^{-1}\\
&\quad=\frac{\varphi(x)-\varphi(y)}{x-y}
\end{aligned}$$

for all $x,y\in W$, where

$$\begin{aligned}
&A:=d-q,\qquad B:=a-b-c+q,\qquad C:=b+c-a-d,\qquad D:=c-p,\\
&E:=b-a-d+p,\qquad H:=a+d-b-c,\qquad K:=b-a,\\
&M:=b+c-a,\qquad N:=a+d-b.
\end{aligned}$$

Letting $y\in W$ in this formula tend to an $y_0\in(\mathrm{cl}\,W)\setminus W$ we obtain

$$\frac{\varphi(x)-y_0}{x-y_0}=\frac{\varphi(y_0)[D\varphi(y_0)+Ey_0+H]}{(Dy_0-c)\varphi(y_0)+Ey_0^2+Ny_0},\quad x\in W.$$

Since $\varphi(y_0)=y_0$ and $H=N-c$, we hence get

$$\varphi(x)=x,\quad x\in W.$$

This contradiction completes the proof. □

4. A two parameter family of means related to a Cauchy mean

Theorem 2. *Let $f,g:(0,\infty)\to\mathbb{R}$ be differentiable functions, $g'\neq 0$, $\frac{f'}{g'}$ one-to-one, and $a,b>0$. Then*

(1) *the function $\psi_{a,b}:(0,\infty)\to\mathbb{R}$,*

$$\psi_{a,b}(x):=\begin{cases}\frac{f(ax)-f(bx)}{g(ax)-g(bx)}, & a\neq b,\\ \frac{f'(ax)}{g'(ax)}, & a=b,\end{cases}\tag{6}$$

is continuous and strictly monotonic.

The function $D_{a,b}^{[f,g]} : (0,\infty)^2 \to (0,\infty)$ *given by*

$$D_{a,b}^{[f,g]}(x,y) := \begin{cases} \psi_{a,b}^{-1}\left(\frac{f(ax)-f(by)}{g(ax)-g(by)}\right), & y \neq \frac{a}{b}x, \\ \psi_{a,b}^{-1}\left(\frac{f'(ax)}{g'(ax)}\right), & y = \frac{a}{b}x, \end{cases} \tag{7}$$

is a strict and increasing mean and

$$D_{1,1}^{[f,g]} = D^{[f,g]};$$

(2) *the following facts are equivalent*:
  (i) $D_{a,b}^{[f,g]}$ *is symmetric*;
  (ii) $D_{a,b}^{[f,g]} = D_{b,a}^{[f,g]}$;
  (iii) $a = b$;

(3) *the mean* $D_{a,b}^{[g,f]}$ *is well defined,*

$$D_{a,b}^{[g,f]} = D_{a,b}^{[f,g]},$$

and $D_{a,a}^{[f,g]}$ *is* φ*-conjugate of the mean* $D^{[g,f]}$ *with* $\varphi(x) = ax$ $(x > 0)$, *that is*

$$D_{a,a}^{[g,f]}(x,y) = \frac{1}{a} D_{a,a}^{[f,g]}(ax,ay), \quad x,y > 0;$$

(4) *for every* $(x,y) \in (0,\infty)^2$, *the function*

$$(0,\infty)^2 \ni (a,b) \to D_{a,b}^{[f,g]}(x,y)$$

is continuous.

Proof. The function g is continuous and strictly monotonic. By Remark 1, the function $\frac{f'}{g'}$ is also continuous and strictly monotonic.

(1) Suppose, for instance, that g and $\frac{f'}{g'}$ are strictly increasing. The identity $\frac{f'}{g'} = (f \circ g^{-1})' \circ g$ implies that the function $f \circ g^{-1}$ is strictly convex. Thus the function

$$g\big((0,\infty)^2\big) \ni (u,v) \to \frac{f(g^{-1}(u)) - f(g^{-1}(v))}{u - v}$$

is strictly increasing with respect to each of the variables. Consequently, the function

$$(0,\infty) \ni (x,y) \to \frac{f(x) - f(y)}{g(x) - g(y)}$$

is strictly increasing with respect to each of the variables. This implies that $\psi_{a,b}$ is strictly increasing. Take $x, y \in (0,\infty)$, $x < y$. If $y \neq \frac{a}{b}x$ then

$$\psi_{a,b}(x) = \frac{f(ax) - f(bx)}{g(ax) - g(bx)} < \frac{f(ax) - f(by)}{g(ax) - g(by)} < \frac{f(ay) - f(bx)}{g(ay) - g(bx)} = \psi_{a,b}(y)$$

whence, as $\psi_{a,b}$ is strictly increasing,

$$\min(x,y) = x < \psi_{a,b}^{-1}\left(\frac{f(ax) - f(by)}{g(ax) - g(by)}\right) < y = \max(x,y).$$

The respective argument in the case $y = \frac{a}{b}x$ is obvious. Since in the remaining three cases with respect to the type of the monotonicity of the functions g and $\frac{f'}{g'}$ the arguments are analogous, we omit them. This proves that $D_{a,b}^{[f,g]}$ is a strict mean which increasing with respect to each of the variables.

(2) The equivalence (i) $\Leftrightarrow$ (ii) is obvious. For an indirect argument of the equivalence (i) $\Leftrightarrow$ (iii), suppose that for some $a, b > 0$, $a \neq b$, the mean $D_{a,b}^{[f,g]}$ is symmetric, that is that

$$D_{a,b}^{[f,g]}(x,y) = D_{a,b}^{[f,g]}(y,x), \quad x, y > 0.$$

This relation holds if and only if

$$\frac{f(ax) - f(by)}{g(ax) - g(by)} = \frac{f(ay) - f(bx)}{g(ay) - g(bx)}, \quad x, y > 0,\ y \neq \frac{a}{b}x,\ y \neq \frac{b}{a}x.$$

Replacing x by $\frac{x}{a}$, y by $\frac{y}{b}$ and setting $c := \frac{b}{a}$ we obtain

$$\frac{f(x) - f(y)}{g(x) - g(y)} = \frac{f(cx) - f(c^{-1}y)}{g(cx) - g(c^{-1}y)}, \quad x, y > 0,\ x \neq y \neq c^2 x.$$

Setting

$$F(u) := f\big(c^u\big), \quad G(u) := g\big(c^u\big), \quad u \in \mathbb{R},$$

we can write this equation in the form

$$\frac{F(u) - F(v)}{G(u) - G(v)} = \frac{F(u+1) - F(v-1)}{G(u+1) - G(v-1)}, \quad u, v \in \mathbb{R},\ u \neq v \neq u+2.$$

Since F and G are continuous and G is strictly monotonic, Theorem 1 implies that $F = AG + B$ for some real constant A, B, $A \neq 0$. Thus $f = Ag + B$ and, consequently, $f'/g' = A$. This contradicts the assumption that f'/g' is one-to-one and completes the proof of part (2).

(3) To show that $D_{a,b}^{[g,f]}$ is well defined and $D_{a,b}^{[g,f]} = D_{a,b}^{[f,g]}$ we can argue as in Remark 2. The linear conjugacy of $D_{a,a}^{[f,g]}$ with the Cauchy mean $D_{a,a}^{[g,f]}$ via the linear function $\varphi(x) = ax$ ($x > 0$), is an immediate consequence of the definition of the mean $D_{a,b}^{[g,f]}$.

Since statement (4) is obvious, the proof is complete. □

Remark 5. We assume that the generators f, g of the means $D_{a,b}^{[f,g]}$ are defined on interval $(0, \infty)$. Obviously, a counterpart of Theorem 2 for $f, g : \mathbb{R} \to \mathbb{R}$ remains true and supply us with a family of means $\{D_{a,b}^{[f,g]}\colon a, b > 0\}$ defined on $\mathbb{R}^2$.

Remark 6. From Theorem 2(3) we get

$$D_{a,a}^{[f,g]}(ax, ay) = a D_{a,a}^{[f,g]}(x,y), \quad a, x, y > 0.$$

Note that the mean $D_{a,a}^{[g,f]}$ is φ-conjugate of the Lagrangean mean $L^{[f \circ g^{-1}]}$ with $\varphi(x) = ax$ ($x > 0$). Clearly, it is no longer true for $D_{a,b}^{[f,g]}$ with $a \neq b$.

Example 1. For $f(x) = x^p$, $g(x) = x^q$ ($x > 0$), where $pq(p-q) \neq 0$, and $a, b > 0$ we have

$$\psi_{a,b}(x) = \begin{cases} \frac{a^p - b^p}{a^q - b^q} x^{p-q}, & a \neq b, \\ \frac{p}{q}(ax)^{p-q}, & a = b. \end{cases}$$

Applying Theorem 2(1), for $a \neq b$ we obtain

$$D_{a,b}^{[g,f]}(x,y) = \begin{cases} \left(\frac{a^q - b^q}{a^p - b^p} \frac{a^p x^p - b^p y^p}{a^q x^q - b^q y^q}\right)^{1/(p-q)}, & y \neq \frac{a}{b}x, \\ \left(\frac{p}{q} \frac{a^q - b^q}{a^p - b^p}\right)^{1/(p-q)} ax, & y = \frac{a}{b}x, \end{cases}$$

and, for $a = b$,

$$D_{a,a}^{[g,f]}(x,y) = \begin{cases} \left(\frac{q}{p}\frac{x^p - y^p}{x^q - y^q}\right)^{1/(p-q)}, & y \neq x, \\ x, & y = x. \end{cases}$$

In particular, for $p = 2$ and $q = 1$, we hence get

$$\psi_{a,b}(x) = (a+b)x, \quad a, b, x > 0 \text{ (respectively } x \in \mathbb{R})$$

and

$$D_{a,b}^{[f,g]}(x,y) = \frac{a}{a+b}x + \frac{b}{a+b}y, \quad a, b, x, y > 0 \text{ (respectively } x, y \in \mathbb{R}).$$

Example 2. For $f(x) = \exp(x)$, $g(x) = x$ we have $\psi_{a,a}(x) = \exp(ax)$ and

$$D_{a,a}^{[f,g]}(x,y) = \frac{1}{a}\log\left(\frac{e^{ax} - e^{ay}}{x - y}\right).$$

If $a \neq b$, then $\psi_{a,b}(x) = \frac{e^{ax} - e^{bx}}{ax - bx}$, and we do not know the effective formula for $D_{a,b}^{[f,g]}$.

In this connection consider the following:

Remark 7. Suppose that $f, g : (0,\infty) \to \mathbb{R}$ satisfy the assumptions of Theorem 2. It is seen from the definition of $\psi_{a,b}$ that in finding the effective formula for $D_{a,b}^{[f,g]}$, the relation $\psi_{a,b} = \phi(a,b)\psi$, for some functions $\phi : (0,\infty)^2 \to \mathbb{R}$ and $\psi : (0,\infty) \to \mathbb{R}$, can be helpful.

We shall prove:

Proposition 2. *Let $f, g : (0,\infty) \to \mathbb{R}$ satisfy the assumptions of Theorem 2. Suppose that, for some functions $\phi : (0,\infty)^2 \to \mathbb{R}$ and $\psi : (0,\infty) \to \mathbb{R}$,*

$$\psi_{a,b}(x) = \phi(a,b)\psi(x), \quad a, b, x > 0. \tag{8}$$

Then there are $r, q \in \mathbb{R} \setminus \{0\}$ such that

$$f(x) = c_1 x^{r+q} + c_0, \quad g(x) = d_1 x^q + d_0, \quad \psi(x) = \psi(1)x^r, \quad x > 0.$$

Proof. Setting $x = 1$ in (8) and taking into account (6), we get

$$\phi(a,b)\psi(1) = \psi_{a,b}(1) = \frac{f(a) - f(b)}{g(a) - g(b)}, \quad a \neq b.$$

This relation and (8) imply

$$\frac{f(ax) - f(bx)}{g(ax) - g(bx)} = \frac{f(a) - f(b)}{g(a) - g(b)}\frac{\psi(x)}{\psi(1)}, \quad a, b, x > 0,\ a \neq b. \tag{9}$$

Letting here $b \to a$ we obtain

$$\frac{f'(ax)}{g'(ax)} = \frac{f'(a)}{g'(a)}\frac{\psi(x)}{\psi(1)}, \quad a, x > 0. \tag{10}$$

Setting here $a = 1$ gives

$$\frac{\psi(x)}{\psi(1)} = \frac{f'(1)}{g'(1)}\frac{f'(x)}{g'(x)}, \quad x > 0,$$

whence, by (10),

$$\frac{g'(1)}{f'(1)}\frac{f'(ax)}{g'(ax)} = \left(\frac{g'(1)}{f'(1)}\frac{f'(a)}{g'(a)}\right)\left(\frac{g'(1)}{f'(1)}\frac{f'(x)}{g'(x)}\right), \quad a, x > 0,$$

which proves that the continuous function $\frac{g'(1)}{f'(1)}\frac{f'}{g'}$ is multiplicative. Thus there is an $r \in \mathbb{R}$ such that

$$\frac{f'(x)}{g'(x)} = \frac{f'(1)}{g'(1)}x^r, \quad x > 0.$$

Of course $r \neq 0$ (in the opposite case the function $\frac{f'}{g'}$ would not be one-to-one). It follows that

$$f'(x) = cx^r g'(x), \quad x > 0; \quad \text{where } c := \frac{f'(1)}{g'(1)}, \tag{11}$$

and

$$\psi(x) = \psi(1)x^r, \quad x > 0. \tag{12}$$

From (9) and (12) we obtain

$$\big[g(a) - g(b)\big]\big[f(ax) - f(bx)\big] = x^r\big[f(a) - f(b)\big]\big[g(ax) - g(bx)\big], \quad a, b, x > 0.$$

Differentiation of both sides, first with respect to a and then the resulting equation with respect to b, gives

$$g'(a)f'(bx) + g'(b)f'(ax) = x^r\big[f'(a)g'(bx) + f'(b)g'(ax)\big], \quad a, b, x > 0.$$

Hence, making use of (11), we get

$$cg'(a)(bx)^r g'(bx) + cg'(b)(ax)^r g'(ax) = x^r\big[ca^r g'(a)g'(bx) + cb^r g'(b)g'(ax)\big]$$

for all $a, b, x > 0$. Since $c \neq 0$, we can write this equation in the form

$$g'(a)g'(bx)\big[(bx)^r - (ax)^r\big] = g'(b)g'(ax)\big[(bx)^r - (ax)^r\big], \quad a, b, x > 0,$$

whence

$$g'(a)g'(bx) = g'(b)g'(ax), \quad a, b, x > 0,\ a \neq b.$$

The continuity of g' implies that

$$g'(a)g'(bx) = g'(b)g'(ax), \quad a, b, x > 0.$$

Setting $b = 1$ gives

$$g'(a)g'(x) = g'(1)g'(ax), \quad a, x > 0,$$

which means that the function $\frac{g'}{g'(1)}$ is multiplicative. Consequently, there exists a $q \in \mathbb{R}$, $q \neq 1$, such that

$$g'(x) = g'(1)x^{q-1}, \quad x > 0,$$

whence

$$g(x) = d_1x^q + d_0, \quad x > 0,$$

for some $d_1, d_0 \in \mathbb{R}$, $d_1 \neq 0$. Now from (11) we get

$$f(x) = c_1x^{r+q} + c_0, \quad x > 0,$$

for some $c_1, c_0 \in \mathbb{R}$, $c_1 \neq 0$. $\square$

Remark 8. It is easy to see that the mean $D_{a,b}^{[f,g]}$ with the functions f, g described in Proposition 2, with $p := q + r$ and such that $pq(p-q) \neq 0$, coincide with those given in Example 1, and we have

$$D_{a,b}^{[f,g]}(x,y) = D_{\frac{a}{b},1}^{[f,g]}(x,y), \quad a,b,x,y > 0.$$

Thus $D_{a,b}^{[f,g]}$ depends only on the parameter $t = \frac{a}{b}$.

We shall use this remark in the next section.

5. An extension of the Stolarsky means

Applying Theorem 2 for functions $f, g : (0,\infty) \to \mathbb{R}$, such that

$$f(x) = x^p, \quad g(x) = x^q \quad (x > 0), \; pq(p-q) \neq 0,$$

taking into account Remark 8, and then making some natural calculations in the case when $pq(p-q) = 0$ (cf. Leach and Sholander [2]), we obtain the following:

Theorem 3. *For every $p, q \in \mathbb{R}$ and $t > 0$ the function $E_t^{[p,q]} : (0,\infty)^2 \to (0,\infty)$ defined by the following formulas*:

$$E_t^{[p,q]}(x,y) := \left(\frac{t^q - 1}{t^p - 1}\frac{t^p x^p - y^p}{t^q x^q - y^q}\right)^{\frac{1}{p-q}}, \quad pq(p-q) \neq 0, \; t > 0, \; t \neq 1,$$

$$E_1^{[p,q]}(x,y) := \left(\frac{q}{p}\frac{x^p - y^p}{x^q - y^q}\right)^{\frac{1}{p-q}}, \quad pq(p-q) \neq 0,$$

$$E_t^{[p,0]}(x,y) = E_t^{[0,p]}(x,y) := \left(\frac{\log t}{t^p - 1}\frac{t^p x^p - y^p}{\log tx - \log y}\right)^{\frac{1}{p}}, \quad p \neq 0, \; t > 0, \; t \neq 1,$$

$$E_1^{[p,0]}(x,y) = E_1^{[0,p]}(x,y) = \left(\frac{1}{p}\frac{x - y}{\log x - \log y}\right)^{\frac{1}{p}}, \quad p \neq 0,$$

$$E_t^{[p,p]}(x,y) = \exp\left(\frac{t^p}{1 - t^p}\right)\left(\frac{(tx)^{(tx)^p}}{y^{y^p}}\right)^{\frac{1}{(tx)^p - y^p}}, \quad p \neq 0, \; t > 0, \; t \neq 1,$$

$$E_1^{[p,p]}(x,y) = \exp\left(-\frac{1}{p}\right) \cdot \left(\frac{y^{y^p}}{x^{x^p}}\right)^{\frac{1}{y^p - x^p}}, \quad q = p \neq 0,$$

$$E_t^{[0,0]}(x,y) = \sqrt{xy}, \quad q = p = 0, \; t > 0,$$

is a strict, increasing and homogeneous mean.

Moreover,

(1) *for all $p, q \in \mathbb{R}$ and $t > 0$,*

$$E_t^{[p,q]} \text{ is symmetric} \quad \text{iff} \quad t = 1; \quad E_t^{[p,q]} = E_t^{[q,p]},$$

and

$$E_{1/t}^{[p,q]}(x,y) = E_t^{[p,q]}(y,x); \quad x,y > 0;$$

(2) *for every* $x, y > 0$, *the function*

$$(0,\infty)^3 \ni (p,q,t) \to E_t^{[p,q]}(x,y)$$

is continuous;

(3) *if* $p^2+q^2>0$ *then, for all* $x, y > 0$,

$$\lim_{t\to 0+} E_t^{[p,q]}(x,y) = \lim_{t\to\infty} E_t^{[p,q]}(y,x) = \begin{cases} y, & p \geqslant 0,\ q \geqslant 0, \\ x^{\frac{-q}{p-q}} y^{\frac{p}{p-q}}, & p>0,\ q<0, \\ x^{\frac{-p}{-p+q}} y^{\frac{q}{-p+q}}, & p<0,\ q>0, \\ x, & p \leqslant 0,\ q \leqslant 0. \end{cases}$$

Remark 9. Since $\{E_1^{[p,q]}\colon\ p,q\in\mathbb{R}\}$ contains the family of Stolarsky means [6] (cf. Leach and Sholander [2]), the family $\{E_t^{[p,q]}\colon\ p,q\in\mathbb{R},\ t>0\}$ is a natural extension of Stolarsky means. In particular, the means $E_t^{[p,0]}$ and $E_t^{[0,p]}$ generalize the logarithmic means and $E_t^{[p,p]}$ the identric means.

Remark 10. Note that $E_t^{[0,0]}$, for all $t>0$, coincides with the symmetric geometric mean $G(x,y) := \sqrt{xy}$ and does not depend on t.

The family $\{E_1^{[p,q]}\colon\ p,q\in\mathbb{R}\}$ can be also obtained from the following:

Proposition 3. *Let* $k\in\mathbb{N}$, $k\geqslant 2$, *be fixed. Suppose that* $M:(0,\infty)^k\to(0,\infty)$ *is a mean. If* M *homogeneous and increasing with respect to each of the variables, then, for every* $w_1,\dots,w_k>0$, *the function* $M_{w_1,\dots,w_k}:(0,\infty)^k\to(0,\infty)$ *defined by*

$$M_{w_1,\dots,w_k}(x_1,\dots,x_k) := \frac{M(w_1x_1,\dots,w_kx_k)}{M(w_1,\dots,w_k)}, \quad x_1,\dots,x_k>0,$$

is an increasing and homogeneous mean in $(0,\infty)$.

Proof. Let us fix $w_1,\dots,w_k>0$. Clearly $M_{w_1,\dots,w_k}$ is increasing with respect to each of the variables and homogeneous. From the homogeneity of M we have

$$M_{w_1,\dots,w_k}(x,\dots,x) := \frac{M(w_1x,\dots,w_kx)}{M(w_1,\dots,w_k)} = x\frac{M(w_1,\dots,w_k)}{M(w_1,\dots,w_k)} = x, \quad x>0,$$

whence, making use of the increasing monotonicity of $M_{w_1,\dots,w_k}$,

$$\min(x_1,\dots,x_k) \leqslant M_{w_1,\dots,w_k}(x_1,\dots,x_k) \leqslant \max(x_1,\dots,x_k), \quad x_1,\dots,x_k>0. \qquad \square$$

6. When $E_t^{[p,q]}$ is a weighted quasi-arithmetic mean

Let $\varphi:(0,\infty)\to\mathbb{R}$ be continuous and strictly monotonic and $w\in(0,1)$ a fixed number. Then $A_w^{[\varphi]}:(0,\infty)\times(0,\infty)\to(0,\infty)$ defined by

$$A_w^{[\varphi]}(x,y) := \varphi^{-1}\big(w\varphi(x)+(1-w)\varphi(y)\big), \quad x,y>0,$$

is a strict continuous mean and it called *weighted quasi-arithmetic*. The function φ is referred to as a *generator* of the mean and w as its *weight*.

Remark 11. It is well known that $A_w^{[\varphi]}$ is homogeneous iff it is a weighted *power mean that* is iff there exists a $q \in \mathbb{R}$ such that $A_w^{[\varphi]} = A_{[q],w}$ where

$$A_{[q],w}(x,y) := \begin{cases} (wx^q + (1-w)y^q)^{1/q} & \text{if } q \neq 0, \\ x^w y^{1-w} & \text{if } q = 0, \end{cases} \qquad x, y > 0.$$

The proof is due to Jessen (cf. Hardy et al. [1, p. 68]).

Theorem 4. *Let $p, q \in \mathbb{R}$ and $t > 0$ and be fixed. The following conditions are equivalent*:

(1) $E_t^{[p,q]}$ *is a weighted quasi-arithmetic mean*;
(2) $E_t^{[p,q]}$ *is a weighted power mean*;
(3) $p = 2q$.

Moreover,

$$E_t^{[2q,q]} = A_{[q], \frac{t^q}{1+t^q}}.$$

Proof. Implication (1) $\Rightarrow$ (2) follows from the homogeneity of $E_t^{[p,q]}$ and Remark 11.

To prove that (2) $\Rightarrow$ (3) assume first that $pq(p-q) \neq 0$ and $t \neq 1$. In this case equality $E_t^{[p,q]} = A_w^{[r]}$ for some $r \in \mathbb{R}$ and $w \in (0,1)$ becomes

$$\left(\frac{t^q - 1}{t^p - 1} \frac{t^p x^p - y^p}{t^q x^q - y^q} \right)^{\frac{1}{p-q}} = \left(wx^r + (1-w)y^r \right)^{1/r}, \qquad x, y > 0.$$

Setting here $y = 1$ we infer that the function

$$x \to \frac{\left(\frac{t^p x^p - 1}{t^q x^q - 1}\right)^{\frac{1}{p-q}}}{(wx^r + 1 - w)^{1/r}}$$

is constant. Calculating the first derivative (with respect to x) we hence get

$$wx^r \left[q(tx)^p - p(tx)^q + p - q \right] - (1-w) \left\{ \left[(p-q)(tx)^q - p \right] (tx)^p + q(tx)^q \right\} = 0$$

that is

$$\frac{x^r [q(tx)^p - p(tx)^q + p - q]}{[(p-q)(tx)^q - p](tx)^p + q(tx)^q} = \frac{1-w}{w}, \qquad x > 0,\ x \neq 1.$$

Multiplying both sides by t^r we get

$$\frac{(tx)^r [q(tx)^p - p(tx)^q + p - q]}{[(p-q)(tx)^q - p](tx)^p + q(tx)^q} = t^r \frac{1-w}{w}, \qquad x > 0,\ x \neq 1.$$

Replacing here x by x/t and y by y/t we obtain

$$x^r \frac{qx^p - px^q + p - q}{(p-q)x^{p+q} - px^p + qx^q} = t^r \frac{1-w}{w}, \qquad x > 0,\ x \neq t,$$

which proves that the left-hand side is constant with respect to x. Now simple calculations show that it can happen if and only if

$$q = r \quad \text{and} \quad p = 2r.$$

Setting $q = r$ and $p = 2r$ into the above equation we obtain, which implies that

$$w = \frac{t^q}{1 + t^q}.$$

In the case $pq(p - q) = 0$, $p^2 + q^2 \neq 0$, and $t \neq 1$, none of the means $E_t^{[p,q]}$ coincides with a power one (cf. the forms of these means in Theorem 3). In the case when $t = 1$ the result is well known.

Since implication (3) $\Rightarrow$ (1) as well as the "moreover" part is obvious, the proof is completed. $\square$

7. Closing remark

Under some strong regularity assumption, Losonczi [3] solved the functional equation $D^{[f,g]} = D^{[F,G]}$ with four unknown function. In [4] we have proved that the regularity assumption can be done without any loss of generality. This was a solution of Problem 7 presented in a survey paper by Páles [5]. The relevant equality problem of the generalized Cauchy means $D_{a.b}^{[f,g]} = D_{c,d}^{[F,G]}$ seems to be more difficult.

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